Chapter 5

Conservation of Mass and Momentum

This chapter derives the basic conservation of mass and momentum laws for a fluid in both integral and differential form. Because of their simplicity and common occurrence in nature, our primary focus will be on incompressible flows. Mass- and momentum-conservation laws provide a sufficient number of equations to solve incompressible-flow problems for which temperature and heat transfer are of no interest. If thermal considerations are required or if viscous losses are important, we require an additional equation based on conservation of energy. We defer development of the energy-conservation law to Chapter 7.

Applying the definition of a system, Newton's second law of motion and the Reynolds Transport Theorem, we develop the mass- and momentum-conservation laws for a finite-sized control volume. Application of the conservation laws in their integral form constitutes a useful tool for analysis of complex fluid-flow problems that warrants detailed analysis and discussion. We present complete details on application of the integral forms in Chapter 6, where we develop what is known as the *control-volume method*.

There is useful insight that can be gleaned from the differential forms of the mass- and momentum-conservation equations, which can be deduced by considering a differential-sized control volume. Thus, after deriving the differential forms, we examine a few properties of fluid motion, including one of the most famous results of fluid mechanics known as **Bernoulli's equation**. This result, derived from the momentum equation, serves as a conservation of mechanical energy law, valid for incompressible flows under a set of commonly-observed constraints. It relates pressure, velocity and potential energy in a moving fluid, and simplifies to the hydrostatic relation when the velocity is zero. Like the hydrostatic relation, Bernoulli's equation serves as the principle upon which important measurement devices are based. We will learn of two such devices known as the **Pitot tube** and the **Pitot-static tube**.

We also demonstrate **Galilean invariance** of the equations of motion. This is a nontrivial consideration for two key reasons. First, a Galilean transformation is a linear operation while the Eulerian description makes the momentum equation quasi-linear. Thus, it is not obvious that the fluid mechanics equations of motion are invariant under a Galilean transformation. Second, if our equations are not invariant, we cannot use measurements in a wind tunnel for a stationary model to infer forces on a prototype moving into a fluid that is at rest. This would greatly complicate the job of the experimenter who would have to design models that are in motion in a wind tunnel.

Bernoulli's equation and Galilean invariance also prove helpful in applying the integral-conservation laws for certain applications. This is the primary reason we pause and develop both the integral and differential forms of the conservation principles prior to developing the *control-volume method* in Chapter 6.

We derived the Reynolds Transport Theorem in Chapter 4 to bridge the gap between the Lagrangian and Eulerian descriptions. It is now possible to apply the familiar conservation laws of classical physics to a volume that contains different fluid particles at each instant. To derive equations expressing conservation of mass, momentum and energy, we appeal to the definition of a system, Newton's second law of motion and the first law of thermodynamics, respectively.

In terms of the nomenclature introduced for the Reynolds Transport Theorem development, we work with corresponding extensive (B) and intensive (β) variable pairs listed in Table 5.1. Note that in the case of the energy-conservation principle, which we have deferred to Chapter 7, the intensive variable is the sum of the internal energy, e, kinetic energy, $\frac{1}{2}\mathbf{u}\cdot\mathbf{u}$, and the body-force potential, $\mathcal{V}$—all per unit mass.

Table 5.1: *Extensive and Intensive Variable Pairs in Conservation Principles*

Property	B	β	Foundation
Mass	M = Mass	1	Definition of a System
Momentum	$\mathbf{P}$ = Momentum	$\mathbf{u}$	Newton's Second Law of Motion
Energy	E = Total Energy	$e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \mathcal{V}$	First Law of Thermodynamics

5.1 Conservation of Mass

In this section, we first derive the mass-conservation law for a finite-sized control volume. Then, we focus on an infinitesimal control volume to deduce the differential equation for mass conservation. To develop these equations, we appeal to the definition of a system.

5.1.1 Integral Form

Consider a general control volume, V, bounded by a closed surface S. Figure 5.1 illustrates such a volume, including a differential volume element, dV, a differential surface element, dS, and an **outer unit normal** vector, $\mathbf{n}$. For the sake of generality, we assume the control volume is moving and denote the bounding-surface velocity by $\mathbf{u}^{cv}$. The fluid velocity is $\mathbf{u}$. Since the total mass in the volume is given by

$$M = \iiint_V \rho \, dV \tag{5.1}$$

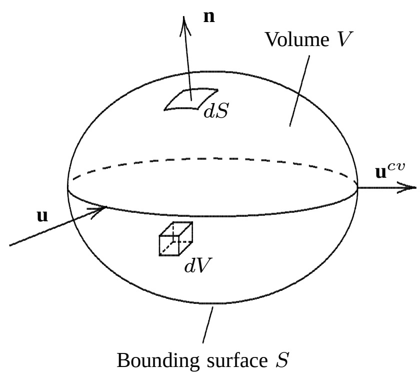


Figure 5.1: *A general control volume for mass conservation.*

the appropriate extensive/intensive variables are $B = M$ and $\beta = 1$ [see Equation (4.50)]. By definition, a *system* always contains the same collection of fluid particles. Consequently, its mass is constant for all time. Thus, the rate of change of the system's mass is zero, which means

$$\frac{dM}{dt} = 0 \tag{5.2}$$

Therefore, invoking the Reynolds Transport Theorem, we arrive at the integral form of the mass-conservation principle for a *control volume*.

$$\frac{d}{dt}\iiint_V \rho\, dV + \oiint_S \rho\, \mathbf{u}^{rel}\cdot \mathbf{n}\, dS = 0 \tag{5.3}$$

where $\mathbf{u}^{rel}$ is the flow velocity relative to the control-volume velocity on the bounding surface, viz.,

$$\mathbf{u}^{rel} \equiv \mathbf{u} - \mathbf{u}^{cv} \tag{5.4}$$

The first term on the left-hand side of Equation (5.3) represents the *instantaneous rate of change of mass in the control volume*. The second term represents the *net flux of mass out of the control volume*.

5.1.2 Differential Form

We deduce a differential equation for mass conservation by applying the limiting form of the Reynolds Transport Theorem, Equation (4.86), to an infinitesimal control volume. Just as in our derivation for a finite-sized control volume, since the mass of a system and the intensive variable β are both constant, necessarily $dB/dt = 0$ and $d\beta/dt = 0$. Hence, inspection of Equation (4.86) shows that the differential equation governing mass conservation for a fluid *at every point within the control volume* is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u}) = 0 \tag{5.5}$$

This equation is often referred to as the **continuity equation**. Equation (5.5) is in what is known as **conservation form**. By definition, this means the differential equation consists of the sum of the time derivative of one quantity and the divergence of another. We can expand $\nabla\cdot(\rho\,\mathbf{u})$ and rewrite Equation (5.5) as

$$\frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0 \tag{5.6}$$

The sum of the first two terms on the left-hand side of Equation (5.6) is the Eulerian derivative of ρ. Thus, we arrive at the continuity equation in **primitive-variable form**.

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{5.7}$$

Finally, note that as mentioned at the end of Subsection 4.7.2, substituting the continuity equation in conservation form [Equation (5.5)] into the Reynolds Transport Theorem as stated in Equation (4.86) yields the simpler Equation (4.87).

5.2 Conservation of Momentum

For simplicity, we consider an inviscid, or perfect, fluid so that only pressure acts on any surface. We will address viscous effects briefly in Chapter 10, and in more complete detail in Chapters 13 and 14.

5.2.1 Integral Form

Letting **P** denote the momentum vector, the momentum of the control volume shown in Figure 5.2 is

$$\mathbf{P} = \iiint_V \rho\,\mathbf{u}\,dV \tag{5.8}$$

so that our extensive variable is **P**, while the intensive variable is **u**. Now, we know from Newton's second law of motion applied to the system coincident with the control volume at an instant in time that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}_s + \mathbf{F}_b \tag{5.9}$$

where $\mathbf{F}_s$ and $\mathbf{F}_b$ denote **surface force** and **body force** exerted by the surroundings on the system, respectively. We define a surface force as one that is transmitted across the surface

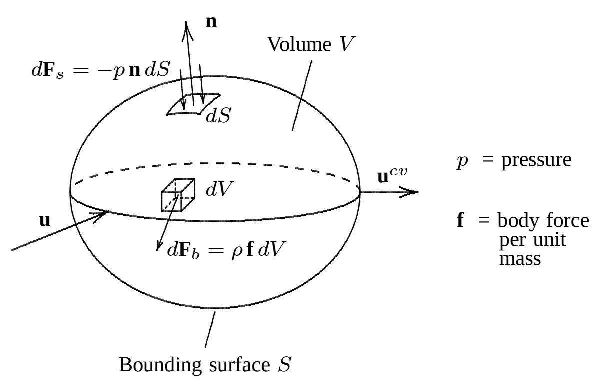


Figure 5.2: *A general control volume for momentum conservation.*

bounding the system. By contrast, a body force acts at a distance, the most common examples being gravitational, electrical and magnetic forces.

Because we have confined our focus to a perfect fluid, the only surface force acting is the fluid pressure. Since **n** is an outer unit normal, the pressure force exerted by the surroundings on a differential surface element dS is $-p\,\mathbf{n}\,dS$. Hence, the net surface force due to pressure imposed by the surroundings on the system is

$$\mathbf{F}_s = -\oiint_S p\,\mathbf{n}\,dS \tag{5.10}$$

As discussed in Chapter 3, this would be the buoyancy force acting on an object submerged or floating in a liquid. Turning to the body force, we introduce the **specific body force vector**, **f**, whose dimensions are force per unit mass. The net body force on the system is given by the following volume integral.

$$\mathbf{F}_b = \iiint_V \rho\,\mathbf{f}\,dV \tag{5.11}$$

As an example, for gravity we would say that $\mathbf{f} = \mathbf{g} = -g\mathbf{k}$, where g = 32.174 ft/sec^2 (9.807 m/sec^2) is the acceleration due to gravity. In this case, the force $\mathbf{F}_b$ would be the control volume's weight.

Thus, invoking the Reynolds Transport Theorem, we arrive at the conservation of momentum principle for a control volume, viz.,

$$\frac{d}{dt}\iiint_V \rho\,\mathbf{u}\,dV + \oiint_S \rho\,\mathbf{u}\,(\mathbf{u}^{rel}\cdot\mathbf{n})\,dS = -\oiint_S p\,\mathbf{n}\,dS + \iiint_V \rho\,\mathbf{f}\,dV \tag{5.12}$$

The first term on the left-hand side of Equation (5.12) represents the *instantaneous rate of change of momentum in the control volume*. The second term represents the *net flux of momentum out of the control volume*. The two terms on the right-hand side are the *net pressure force* and *net body force* exerted by the surroundings on the control volume.

Note that, consistent with Newton's laws of motion, Equation (5.12) is a conservation law for **absolute momentum**, $\rho\,\mathbf{u}$, not for momentum relative to the control volume. Hence, $\mathbf{u}^{rel}$ appears only in the surface-flux integral [the second term on the left-hand side of Equation (5.12)]. We will discuss the issue of absolute momentum in greater detail when we focus on accelerating control volumes in Section 6.5.

5.2.2 Differential Form

We deduce a differential equation for momentum conservation by applying the limiting form of the Reynolds Transport Theorem, Equation (4.87), to an infinitesimal control volume. As with a finite-sized control volume, we begin with Newton's second law of motion, viz.,

$$\frac{d\mathbf{P}}{dt} = -\oiint_S p\,\mathbf{n}\,dS + \iiint_V \rho\,\mathbf{f}\,dV \tag{5.13}$$

where **f** is the specific body force vector. Proceeding term by term from left to right, the Reynolds Transport Theorem [Equation (4.87)] tells us that for the momentum equation,

$$\lim_{\Delta V\to 0}\frac{1}{\Delta V}\frac{d\mathbf{P}}{dt} = \rho\frac{d\mathbf{u}}{dt} \tag{5.14}$$

Next, recall that in Section 3.1 we evaluated the net pressure force acting on an infinitesimal control volume and demonstrated that

$$-\oiint_S p\,\mathbf{n}\,dS \approx -\nabla p\,\Delta V \quad \text{for} \quad \Delta V \to 0 \quad \Longrightarrow \quad -\lim_{\Delta V\to 0} \frac{1}{\Delta V} \oiint_S p\,\mathbf{n}\,dS = -\nabla p \tag{5.15}$$

Also, for the obvious reason,

$$\lim_{\Delta V\to 0} \frac{1}{\Delta V} \iiint_V \rho\,\mathbf{f}\,dV = \rho\,\mathbf{f} \tag{5.16}$$

Collecting all of this, the resulting differential equation governing momentum conservation at each point in a flowfield is

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\,\mathbf{f} \tag{5.17}$$

This equation, valid for a perfect fluid, is known as **Euler's equation**. It is in primitive-variable form, and can be used to compute the details of general fluid motion at every point in a flow. In words, this equation says that *mass per unit volume times acceleration equals the sum of forces per unit volume*, i.e., it is Newton's second law of motion per unit volume.

5.3 Summary of the Conservation Equations

It is worthwhile to summarize the equations developed in this chapter. Considering first the integral conservation forms, the equations of motion for an inviscid fluid are as follows.

Mass Conservation:

$$\frac{d}{dt} \iiint_V \rho\,dV + \oiint_S \rho\,\mathbf{u}^{rel}\cdot\mathbf{n}\,dS = 0 \tag{5.18}$$

Momentum Conservation:

$$\frac{d}{dt} \iiint_V \rho\,\mathbf{u}\,dV + \oiint_S \rho\,\mathbf{u}\,(\mathbf{u}^{rel}\cdot\mathbf{n})\,dS = -\oiint_S p\,\mathbf{n}\,dS + \iiint_V \rho\,\mathbf{f}\,dV \tag{5.19}$$

Equation of State:

$$\rho = \begin{cases} \dfrac{p}{RT}, & \text{gases} \\ \text{constant}, & \text{liquids} \end{cases} \tag{5.20}$$

Turning to the differential forms of the conservation laws, we have deduced the continuity and Euler equations that govern conservation of mass and momentum, respectively.

Continuity:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{5.21}$$

Euler's Equation:

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\,\mathbf{f} \tag{5.22}$$

An excellent exercise in any branch of mathematical physics, or more generally for any mathematics problem, is to count unknowns and equations. Considering liquids first, the unknowns are the density, ρ, pressure, p, and the three velocity components, (u, v, w). Thus, we have a total of five unknowns. Conservation of mass and the equation of state are both scalar equations, while momentum is a vector equation with three components. Thus, we have five equations to solve for five unknowns. Our mathematical system is said to be closed as we have a sufficient number of equations to solve for the unknowns.

Turning to gases, note that the equation of state introduces the temperature as an additional unknown. We thus have six unknowns for a gas. However, conservation of mass, momentum and the state equation still account for only five equations. Our system is not closed as we lack a sufficient number of equations to solve for all of the unknowns.

Actually, we don't have enough equations for a liquid either if the temperature is required, as it would be for a flow with heat transfer. In both cases we must also consider energy conservation in order to completely specify all properties in a given fluid flow. Nevertheless, there is a wide range of problems we can solve without considering energy conservation. Specifically, as long as we confine our attention to incompressible flows without heat transfer, we can treat ρ as a constant. For such flows, we have five equations and five unknowns for both liquids and gases. As we will learn in Chapter 8, variations in the density of a gas are negligible for low-speed flows, i.e., for flows with Mach number less than about 0.3. We will consider energy conservation in Chapter 7.

The integral form of the conservation laws serve as the foundation of the *control-volume method*. Because of its central importance in fluid mechanics, we will examine all of the nuances of the method in Chapter 6. The balance of this chapter will address properties of the differential form of the conservation laws.

5.4 Mass Conservation for Incompressible Flows

In Cartesian coordinates, the continuity equation [Equation (5.21)] is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0 \tag{5.23}$$

Continuity assumes an especially simple form for incompressible flows. Specifically, if the density, ρ, is constant, we have

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Incompressible)} \tag{5.24}$$

Equation (5.24) has the remarkable property that it holds for both steady and unsteady incompressible flows. Given the velocity vector for a flowfield, we can use this equation to determine whether or not the flow is incompressible. We can also use it to determine necessary conditions for a flow to be incompressible. The following two examples illustrate how the incompressible continuity equation can be used.

Example 5.1 *Consider two-dimensional flow approaching a stagnation point. In the immediate vicinity of the stagnation point, the velocity vector is* $\mathbf{u} = A(x\,\mathbf{i} - y\,\mathbf{j})$, *where* x *and* y *are tangent to and normal to the surface, respectively. The quantity* A *is a constant. Is this flow incompressible?*

Solution. Taking the divergence of this vector, we find

$$\nabla \cdot \mathbf{u} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) \cdot (Ax\,\mathbf{i} - Ay\,\mathbf{j}) = A - A = 0$$

Thus, we conclude that this flow is incompressible.

Example 5.2 *An unsteady flow has the following velocity field:*

$$\mathbf{u} = \Omega(\Omega x t + y)\,\mathbf{i} + \Omega(Ax + B\Omega y t)\,\mathbf{j}$$

where Ω *is a constant of dimension 1/time. The quantities* A *and* B *are dimensionless constants. The flow is incompressible and irrotational. Find the values of* A *and* B *necessary to guarantee these conditions.*

Solution. Since the flow is incompressible and two-dimensional, necessarily

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Omega \frac{\partial}{\partial x}(\Omega x t + y) + \Omega \frac{\partial}{\partial y}(Ax + B\Omega y t) = \Omega^2 (t + Bt) = 0 \qquad \Longrightarrow \qquad B = -1$$

Also, since the flow is irrotational, the vorticity is $(\partial v/\partial x - \partial u/\partial y)\mathbf{k} = \mathbf{0}$, so that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Omega \frac{\partial}{\partial x}(Ax + B\Omega y t) - \Omega \frac{\partial}{\partial y}(\Omega x t + y) = \Omega(A - 1) = 0 \qquad \Longrightarrow \qquad A = 1$$

5.5 Euler's Equation

Although Euler's equation [Equation (5.22)] looks relatively simple in vector form, our shorthand notation for the Eulerian derivative conceals the complexity of this vector partial differential equation. Expanding the differential operator d/dt into its unsteady and convective parts, the three components of Euler's equation in Cartesian coordinates are

$$\left.\begin{aligned} \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + \rho f_x \\ \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \rho f_y \\ \rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \rho f_z \end{aligned}\right\} \tag{5.25}$$

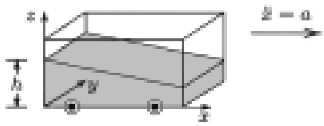
Inspection of Equations (5.25) tells us that Euler's equation is not linear. That is, the convective acceleration terms, i.e., terms such as $u\partial u/\partial x$, involve products of the velocity and its derivative. These terms make the equation **quasi-linear**.[1] Unlike linear equations, we cannot use superposition or even prove that our solution exists and is unique.

On the one hand, these coupled, quasi-linear, partial differential equations are not easy to solve, even for simple, idealized geometries. One noteworthy exception is for incompressible, irrotational flow, known as **potential flow**. As we will see in Chapter 11, under these conditions the continuity equation provides a linear partial differential equation and all of the nonlinearity is confined to the relation between pressure and velocity, viz., through Bernoulli's equation, which we will derive in Section 5.6. Even in this special case, solution of the equations of motion requires complex computer programs for general fluid-flow problems.

On the other hand, after several decades of research and development, such programs are readily available, not only for potential flows, but for rotational flows in a compressible medium. This is one of the key advances in fluid mechanics attributable to the special branch known as Computational Fluid Dynamics (CFD).

[1] We stop short of calling the equation **nonlinear** because, in strict mathematical terms, an equation is nonlinear when the highest derivative in the equation appears in other than a linear form.

Example 5.3 *A rectangular tank of water has constant acceleration a in the x direction. Compute the pressure in the tank, using the fact that $p = p_a$ at $x = 0$ and $z = h$.*



Solution. Euler's equation tells us that $\rho\mathbf{a} = -\nabla p + \rho\mathbf{g}$, which means

$$\rho a = -\frac{\partial p}{\partial x}, \quad 0 = -\frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial p}{\partial z} - \rho g$$

Since $\partial p/\partial y = 0$, the pressure is at most a function of x and z. Integrating first over x,

$$p(x,z) = -\rho a x + f(z)$$

where $f(z)$ is a function of integration. Then, differentiating with respect to z and using the z component of Euler's equation, we find

$$\frac{\partial p}{\partial z} = f'(z) = -\rho g \qquad \Longrightarrow \qquad f(z) = C - \rho g z$$

where C is a constant. Thus, the pressure throughout the fluid in the tank is

$$p(x,z) = C - \rho a x - \rho g z$$

Finally, since $p(0,h) = p_a$, necessarily $C = p_a + \rho g h$, so that the pressure becomes

$$p(x,z) = p_a - \rho a x + \rho g(h - z)$$

Example 5.4 *The average velocity of water flowing through a nozzle increases from $u_1 = 5$ m/sec to $u_2 = 20$ m/sec. Assuming the average velocity varies linearly with distance along the nozzle, x, and that the length of the nozzle is $\ell = 1$ m, estimate the pressure gradient, dp/dx, at a point midway through the nozzle. The density of water is $\rho = 1000$ kg/m^3. You may assume the flow can be approximated as one dimensional.*

Solution. From the one-dimensional Euler equation,

$$\rho u \frac{du}{dx} = -\frac{dp}{dx}$$

The velocity varies linearly from u_1 to u_2 as x increases from 0 to ℓ. Thus,

$$u(x) = u_1 + (u_2 - u_1)\,\frac{x}{\ell} \qquad \Longrightarrow \qquad \frac{du}{dx} = \frac{u_2 - u_1}{\ell}$$

Halfway through the nozzle, we thus have

$$u = \frac{1}{2}\,(u_1 + u_2) \qquad \text{and} \qquad \frac{du}{dx} = \frac{u_2 - u_1}{\ell}$$

Therefore, the pressure gradient is

$$\frac{dp}{dx} = -\rho\left(\frac{u_1 + u_2}{2}\right)\left(\frac{u_2 - u_1}{\ell}\right) = -\rho\frac{u_2^2 - u_1^2}{2\ell}$$

For the given values,

$$\frac{dp}{dx} = -\left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\frac{(20\ \text{m/sec})^2 - (5\ \text{m/sec})^2}{2(1\ \text{m})} = -1.875 \cdot 10^5\ \frac{\text{N}}{\text{m}^3} = -187.5\ \frac{\text{kPa}}{\text{m}}$$

5.5.1 Rotating Tank

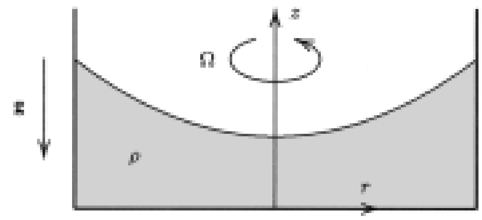


Figure 5.3: *Rotating tank of incompressible fluid—cross-sectional view.*

Incompressible flow in a rotating cylindrical tank (Figure 5.3) is an interesting flow to analyze using Euler's equation. We assume the tank has been rotating for a long time, so that the fluid all moves with constant angular velocity, $\mathbf{\Omega} = \Omega\mathbf{k}$. That is, the fluid is in a state of **rigid-body rotation** (recall our discussion of flow in a rotating cylinder in Section 4.3), so that the velocity at any point in the fluid is

$$\mathbf{u} = \mathbf{\Omega} \times \mathbf{r} = \Omega r \mathbf{e}_\theta \tag{5.26}$$

where r is radial distance from the center of the tank, and $\mathbf{e}_\theta$ is a unit vector in the circumferential direction. Hence, as shown in Section 4.3, the vorticity is

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = 2\mathbf{\Omega} \tag{5.27}$$

Because the velocity is given for this simple example, we can use Euler's equation to solve for the pressure throughout the fluid.

Since the geometry is symmetric about the z axis, we use the axisymmetric form of Euler's equation. From Appendix D, we find that the three components are as follows.

$$\left.\begin{aligned}
\rho u_r \frac{\partial u_r}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \rho w \frac{\partial u_r}{\partial z} - \rho \frac{u_\theta^2}{r} &= -\frac{\partial p}{\partial r} \\
\rho u_r \frac{\partial u_\theta}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \rho w \frac{\partial u_\theta}{\partial z} + \rho \frac{u_r u_\theta}{r} &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\
\rho u_r \frac{\partial w}{\partial r} + \rho \frac{u_\theta}{r}\frac{\partial w}{\partial \theta} + \rho w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} - \rho g
\end{aligned}\right\} \tag{5.28}$$

Now, for rigid-body rotation, we know that the radial (u_r) and axial (w) velocity components vanish and the circumferential component is given by $u_\theta = \Omega r$. Hence, Equations (5.28) simplify to

$$\left.\begin{aligned}
-\rho\Omega^2 r &= -\frac{\partial p}{\partial r} \\
0 &= -\frac{\partial p}{\partial \theta} \\
0 &= -\frac{\partial p}{\partial z} - \rho g
\end{aligned}\right\} \tag{5.29}$$

To solve this coupled set of equations, we begin by integrating the first of the three with respect to r. Note that when we perform an integration of a function of r, θ and z with respect to r, we introduce a *function of integration*, $f(\theta, z)$. This is the analog of the *constant of integration* that appears when we integrate a function of a single variable. Therefore, we have

$$p(r,\theta,z) = \frac{1}{2}\rho\Omega^2 r^2 + f(\theta,z) \tag{5.30}$$

Next, we differentiate Equation (5.30) with respect to θ and substitute into the second of Equations (5.29), viz.,

$$\frac{\partial p}{\partial \theta} = \frac{\partial f}{\partial \theta} = 0 \qquad \Longrightarrow \qquad f(\theta,z) = F(z) \tag{5.31}$$

That is, we have shown that, at most, our function of integration is a function only of z. This is consistent with the axial symmetry of the flow. So, the pressure is now given by

$$p(r,\theta,z) = \frac{1}{2}\rho\Omega^2 r^2 + F(z) \tag{5.32}$$

Finally, to determine the function $F(z)$, we differentiate Equation (5.32) with respect to z and substitute into the last of Equations (5.29). This yields

$$\frac{\partial p}{\partial z} = \frac{dF}{dz} = -\rho g \qquad \Longrightarrow \qquad F(z) = -\rho g z + \text{constant} \tag{5.33}$$

wherefore the pressure is given by

$$p(r,\theta,z) = \frac{1}{2}\rho\Omega^2 r^2 - \rho g z + \text{constant} \tag{5.34}$$

Thus, we conclude that the pressure in the rotating tank satisfies the following equation.

$$p + \rho g z - \frac{1}{2}\rho\Omega^2 r^2 = \text{constant} \tag{5.35}$$

Example 5.5 *Determine the shape of the free surface for the rotating tank of incompressible fluid in Figure 5.3.*

Solution. Denoting the depth of the fluid at the center of the tank (where $r = 0$) by $z = z_{min}$, Equation (5.35) tells us that

$$p + \rho g z - \frac{1}{2}\rho\Omega^2 r^2 = p_a + \rho g z_{min}$$

Since the pressure is atmospheric at the liquid-air interface, i.e., at the free surface, we have

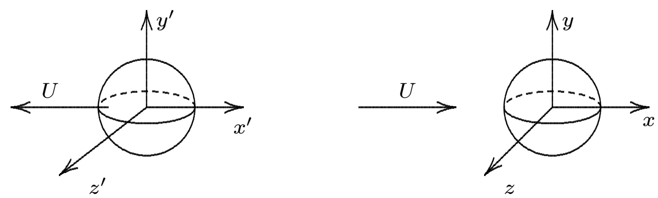
$$p_a + \rho g z - \frac{1}{2}\rho\Omega^2 r^2 = p_a + \rho g z_{min}$$

Simplifying, the shape of the free surface is

$$z = z_{min} + \frac{\Omega^2 r^2}{2g}$$

5.5.2 Galilean Invariance of Euler's Equation

Suppose we have a body advancing into a quiescent fluid with constant velocity $\mathbf{u} = -U\mathbf{i}$ as illustrated in Figure 5.4(a). Clearly, this flow is unsteady for an observer in the main body of fluid since the geometry looks different at each instant. Now, suppose we observe the motion in a coordinate frame translating with the same velocity as the body [Figure 5.4(b)]. In this coordinate frame, the flow geometry does not change with time, and the motion is in fact steady. This is a dramatic improvement from an analytical point of view.



(a) Body in motion (b) Body at rest

Figure 5.4: *Motion of a body in different coordinate frames.*

This is a smooth move provided the Euler equation is invariant under this transformation. What we have done is made the classical **Galilean transformation**. When we say the Euler equation is invariant under such a transformation, we mean the equation holds when we write the equation in terms of all transformed velocities and pertinent flow properties. The physical meaning of invariance is straightforward. From elementary physics we know that Newton's laws of motion are Galilean invariant for motion of discrete particles, and there is no reason for this to change for fluids. However, there is some cause for concern as the Galilean transformation is a linear operation and the Eulerian description introduces nonlinear terms in the time derivative. Hence, the purpose of this section is to demonstrate Galilean invariance of the Euler equation.

Before proceeding to the proof, it is worthwhile to pause and discuss the reason why Galilean invariance matters. Very simply, if the equations of fluid mechanics were not Galilean invariant, any measurements made in a wind tunnel with fluid moving past a stationary model could not be used to predict forces on a full-scale model moving through a fluid at rest. Rather, the model would have to move through the wind tunnel to simulate the full-scale object's motion, with all the difficulties attending acceleration from rest, attainment of steady flow, and deceleration to rest prior to crashing into the end of the tunnel. Clearly, it is preferable to have a stationary model, and Galilean invariance of the equations of motion guarantees applicability of the measurements to a moving object.

Letting primed quantities denote conditions in the frame where the fluid is at rest and the body moves, the Euler equation is given by

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \, \mathbf{u}' = -\frac{\nabla' p}{\rho} + \mathbf{f} \tag{5.36}$$

Clearly, since p and ρ are thermodynamic properties of the fluid, they cannot depend upon the coordinate frame from which we make our observations. Likewise, the body force, $\mathbf{f}$, will be independent of the coordinate frame provided it doesn't depend explicitly upon velocity or position. Body forces that are coordinate-system dependent do exist, e.g., Coriolis force (see

Appendix D, Section D.4), but their presence signifies a noninertial frame for which Galilean invariance does not hold.

By definition, in a Galilean transformation, the coordinates and time transform according to (see Figure 5.4)

$$x = x' + Ut, \quad y = y', \quad z = z', \quad t = t' \tag{5.37}$$

where unprimed quantities correspond to the frame in which the body is at rest. Also, the velocity vectors in the two coordinate frames are related by

$$\mathbf{u} = \mathbf{u}' + U\mathbf{i} \tag{5.38}$$

Clearly, spatial differentiation is unaffected by a Galilean transformation, so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad \Longrightarrow \qquad \nabla = \nabla' \tag{5.39}$$

By contrast, temporal differentiation is affected. From the chain rule, the time derivative transforms as follows.

$$\left(\frac{\partial}{\partial t'}\right)_{x'} = \left(\frac{\partial t}{\partial t'}\right)_{x'} \left(\frac{\partial}{\partial t}\right)_{x} + \left(\frac{\partial x}{\partial t'}\right)_{x'} \left(\frac{\partial}{\partial x}\right)_{t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \tag{5.40}$$

Note that we have omitted y and z from Equation (5.40) for the sake of brevity as all attending derivatives vanish (only x depends upon t). So, transforming the unsteady term first,

$$\frac{\partial \mathbf{u}'}{\partial t'} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\mathbf{u} - U\mathbf{i}) = \frac{\partial \mathbf{u}}{\partial t} + U\frac{\partial \mathbf{u}}{\partial x} \tag{5.41}$$

Thus, the unsteady term is most certainly not invariant under a Galilean transformation.[2] Now consider the convective acceleration. We have

$$\mathbf{u}' \cdot \nabla' \mathbf{u}' = (\mathbf{u} - U\mathbf{i}) \cdot \nabla (\mathbf{u} - U\mathbf{i}) = \mathbf{u} \cdot \nabla \mathbf{u} - U\frac{\partial \mathbf{u}}{\partial x} \tag{5.42}$$

which shows that the convective acceleration is not Galilean invariant either. However, when we sum the unsteady and convective acceleration terms, we find

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = \frac{\partial \mathbf{u}}{\partial t} + U\frac{\partial \mathbf{u}}{\partial x} + \mathbf{u} \cdot \nabla \mathbf{u} - U\frac{\partial \mathbf{u}}{\partial x} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \tag{5.43}$$

In other words, the sum of the unsteady and convective acceleration terms, which is the Eulerian derivative, is invariant under a Galilean transformation. Therefore, the Euler equation in the transformed coordinate frame is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f} \tag{5.44}$$

which is identical to Equation (5.36) with all primes omitted. Thus, the Euler equation is invariant under a Galilean transformation.

[2]As discussed at the beginning of this section, a term is invariant under the transformation if and only if transforming the term is equivalent to dropping primes.

5.6 Bernoulli's Equation

In order to compute flow of an inviscid, or perfect, fluid about a specified object we must solve the continuity equation [Equation (5.21)] and Euler's equation [Equation (5.22)], subject to appropriate boundary conditions. Although closed-form solutions exist for a few simple geometries, general flowfield solutions require carefully formulated computer programs. However, under special conditions, we can integrate the Euler equation to yield a famous result known as **Bernoulli's equation**. To derive it, we will integrate Euler's equation subject to a few limiting conditions, which we will identify below.

Our first step in our derivation is to note that the convective acceleration, $\mathbf{u}\cdot\nabla\mathbf{u}$, which appears in Euler's equation can be rewritten as follows (see Appendix C).

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{5.45}$$

This is a general vector identity that will help us arrive at the desired result with a minimum of algebraic operations. Hence, since total acceleration is the sum of instantaneous and convective contributions, i.e., $d\mathbf{u}/dt = \partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u}$, in terms of vorticity, $\boldsymbol{\omega} = \nabla\times\mathbf{u}$, Euler's equation becomes

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \rho\,\mathbf{u}\times\boldsymbol{\omega} = -\nabla p + \rho\,\mathbf{f} \tag{5.46}$$

As with all of our analysis in this chapter, we presume that the fluid with which we are working is inviscid. We must require four more conditions in order to arrive at this most famous of all equations in the field of fluid mechanics. So, our list of five constraints on the flow under consideration is as follows.

1. Inviscid fluid (Euler's equation is valid);
2. Steady flow ($\partial\mathbf{u}/\partial t = \mathbf{0}$);
3. Incompressible flow (ρ = constant);
4. Conservative body force ($\mathbf{f} = -\nabla\mathcal{V}$);
5. Irrotational flow ($\boldsymbol{\omega} = \mathbf{0}$).

Under these conditions, Equation (5.46) simplifies to

$$\nabla\left(p + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho\mathcal{V}\right) = \mathbf{0} \tag{5.47}$$

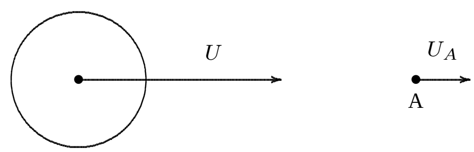
Since this holds for all points in the flow, necessarily the quantity in parentheses is constant. Therefore, we conclude that

$$p + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho\mathcal{V} = \text{constant} \tag{5.48}$$

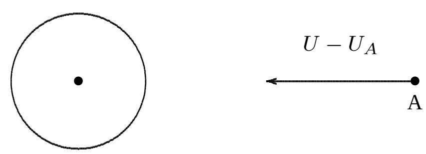
Equation (5.48) is known as **Bernoulli's equation**. Although we arrived at this result by integrating the momentum equation, it is actually an equation for *conservation of mechanical energy*. This is especially clear when the body force is gravity for which the body-force potential is $\mathcal{V} = gz$. Note that in this spirit, we can regard pressure as the pressure-force potential. Equation (5.48) says the sum of the pressure, p, kinetic energy per unit volume (also known as the **dynamic pressure** or **dynamic head**), $\frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u}$, and potential energy per unit volume, $\rho\mathcal{V}$, is constant. There is no contradiction here. A constant-density fluid flow is specified completely by the momentum and continuity equations, so any other property can be deduced from them.

It is always a major simplification if we can use Bernoulli's equation to relate pressure and velocity, rather than having to solve Euler's equation. The reason is obvious, i.e., Bernoulli's equation is a solution to Euler's equation so that no additional computation is needed. With the exception of certain idealized flows, solutions to Euler's equation are difficult to obtain and usually require a numerical solution. For this reason, Bernoulli's equation, whenever valid, provides a major simplification for determining flow properties.

Example 5.6 *A body moves at constant velocity U = 30 ft/sec through water (ρ = 1.94 slug/ft^3). The difference between the pressure at the front stagnation point on the body and at Point A is $p_{stag} - p_A$ = 4.91 psi. What is the velocity at Point A?*



Solution. We must make a Galilean transformation in order to arrive at a steady flow, wherefore we can use Bernoulli's equation. The velocities transform as indicated in the figure below.



By subtracting U from the velocity of the cylinder, it is now at rest. Subtracting U from the velocity at Point A shows that the flow moves to the left at a speed $U - U_A$. Because the flow is steady in this coordinate frame, we can use Bernoulli's equation, viz.,

$$p_{stag} = p_A + \frac{1}{2}\rho\,(U - U_A)^2 \qquad \Longrightarrow \qquad U - U_A = \sqrt{\frac{2}{\rho}\,(p_{stag} - p_A)}$$

Hence, the velocity at Point A is

$$U_A = U - \sqrt{\frac{2}{\rho}\,(p_{stag} - p_A)}$$

For the given values,

$$U_A = 30\ \frac{\text{ft}}{\text{sec}} - \sqrt{\frac{2}{1.94\ \text{slug/ft}^3}\left(4.91\ \frac{\text{lb}}{\text{in}^2}\right)\left(144\ \frac{\text{in}^2}{\text{ft}^2}\right)} = 3\ \frac{\text{ft}}{\text{sec}}$$

Regarding the conditions required for Bernoulli's equation to hold, we can actually relax the irrotationality, incompressibility and steady-flow conditions. Focusing first on the irrotationality constraint, when an incompressible, steady flow with a conservative body force has nonzero vorticity, Equation (5.48) holds along a streamline, although the constant assumes a different value on each streamline. To see this, note first that for steady, incompressible flow with a body-force potential $\mathcal{V}$, Euler's equation simplifies to

$$\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\left(\frac{p}{\rho}+\mathcal{V}\right)=\mathbf{0} \tag{5.49}$$

Using natural coordinates (see Appendix D), we can write the component of Equation (5.49) along a streamline as

$$u\frac{\partial u}{\partial s}+\frac{\partial}{\partial s}\left(\frac{p}{\rho}+\mathcal{V}\right)=0 \qquad \Longrightarrow \qquad \frac{\partial}{\partial s}\left(\frac{p}{\rho}+\frac{1}{2}u^2+\mathcal{V}\right)=0 \tag{5.50}$$

where we have used the fact that $u\partial u/\partial s=\partial(\frac{1}{2}u^2)/\partial s$. This shows that the quantity in parentheses is constant only along a streamline as opposed to being constant everywhere. By contrast, we found above that when the flow is irrotational, the gradient of this quantity [cf. Equation (5.47)] vanishes, which is a much stronger condition. Nevertheless, we conclude that even when the flow has nonzero vorticity,

$$p+\frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u}+\rho\mathcal{V}=\text{constant on streamlines for }\nabla\times\mathbf{u}\neq\mathbf{0} \tag{5.51}$$

where we note that $u^2=\mathbf{u}\cdot\mathbf{u}$. This result is interesting from a conceptual point of view, but is not very helpful in general practice. That is, we don't know where the streamlines are until we have solved the equations of motion. Hence, this form of Bernoulli's equation is more limited than the form for irrotational flow.

Example 5.7 *Determine the pressure in the rotating tank of incompressible fluid in Figure 5.3.*

Solution. As we saw in Subsection 5.5.1, this flow is rotational for a stationary observer and that precludes using Bernoulli's equation. However, the fluid is at rest for an observer rotating with the tank. Thus, the flow in a tank-fixed rotating coordinate frame is steady, irrotational and we are given that the fluid is incompressible. If we can show that the body forces acting are conservative, we can apply Bernoulli's equation in this coordinate frame.

There are two body forces acting, namely, the gravitational force and the centrifugal force. We already know that gravity is a conservative body force and its force potential function is $\mathcal{V}_g=gz$. By definition, the centrifugal force in a coordinate frame rotating with circumferential velocity $u_\theta=\Omega r$ is

$$\mathbf{f}_c=\frac{u_\theta^2}{r}\,\mathbf{e}_r=\Omega^2 r\,\mathbf{e}_r$$

where r is radial distance in a horizontal plane measured from the center of the tank and $\mathbf{e}_r$ is a unit vector pointing radially outward. Therefore,

$$\mathbf{f}_c=-\nabla\mathcal{V}_c=-\frac{\partial\mathcal{V}_c}{\partial r}\,\mathbf{e}_r \quad \text{where} \quad \mathcal{V}_c=-\frac{1}{2}\Omega^2 r^2$$

So, since both of the body forces acting are conservative and the other necessary conditions are satisfied, an observer rotating with the tank can use Bernoulli's equation. Consequently, the pressure is given by

$$p+\rho\mathcal{V}_g+\rho\mathcal{V}_c=\text{constant}$$

Substituting for the force potentials, we conclude that

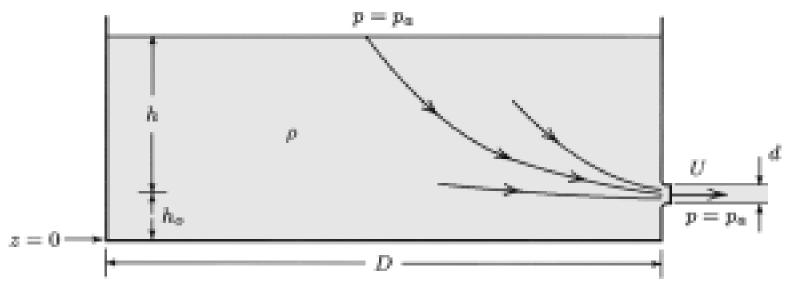
$$p+\rho\,gz-\frac{1}{2}\rho\,\Omega^2 r^2=\text{constant}$$

Under certain conditions, we can relax the incompressibility constraint. Specifically, a modified version of Bernoulli's equation exists for what is known as a **barotropic fluid**, i.e., a fluid for which pressure is strictly a function of density. This is not a pathological case. For example, we will see in Chapter 7 that inviscid flow of a perfect gas with no heat transfer has $p = A\rho^\gamma$ where A is a constant. A straightforward exercise (see Problems section) shows that the modified Bernoulli's equation is

$$\int \frac{dp}{\rho} + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho\mathcal{V} = \text{constant}, \qquad \text{Barotropic fluid} : p = p(\rho) \tag{5.52}$$

Finally, if a flow is incompressible and irrotational, we can develop an unsteady-flow version of Bernoulli's equation. We will derive and make use of this form in Chapter 11 (see also Problems section). The unsteady version permits developing closed-form solutions for simple geometries that are accelerating. The following problem illustrates the use of Bernoulli's equation in a **quasi-steady** flow, i.e., a flow in which properties are changing so slowly in time that it can be treated as though it is steady.

Example 5.8 *Consider a large tank with a small hole a distance h below the surface. A jet of fluid issues from the hole with velocity U. You may assume that, as is generally true for thin jets, the surrounding air impresses atmospheric pressure, p_a, throughout the jet. Assuming the tank diameter, D, is very large compared to the diameter of the small hole, d, determine U as a function of g and h.*



Solution. Bernoulli's equation tells us that

$$p + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho g z = \text{constant}$$

The tank is open to the atmosphere so that the pressure at the top of the tank is also p_a. Since the tank diameter, D, is very large compared to the diameter of the small hole, d, we can ignore the velocity of the fluid at the top of the tank. Hence, we can use Bernoulli's equation to relate a point at the top of the tank to a point in the jet to show that

$$p_a + \rho g(h + h_o) = p_a + \frac{1}{2}\rho U^2 + \rho g h_o$$

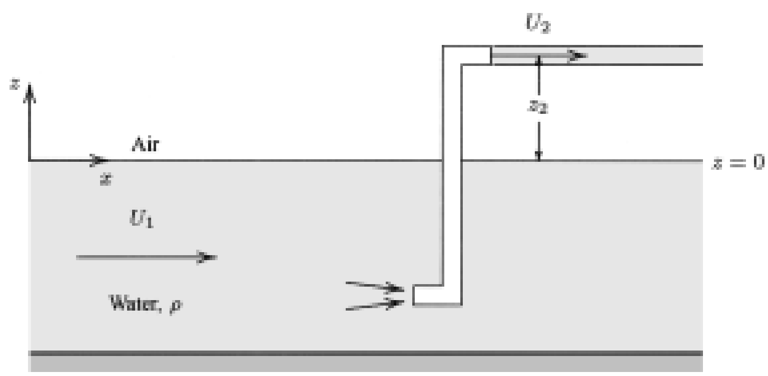
Note that we can set the origin, $z = 0$, anywhere we wish. This is true since the potential energy appears on both sides of this equation, so that only the difference in potential energy between the surface and the jet, $\rho g h$, matters. Thus, the velocity in the jet is given by

$$U = \sqrt{2gh}$$

In Example 5.8, we used Bernoulli's equation to relate conditions at two specified points in the flow. Often it is helpful to determine the "constant" in Bernoulli's equation by evaluating each term at a point in the flow where all terms are known. Typically, we seek a point that lies very far from solid boundaries where the flow is uniform. The most important point about using Bernoulli's equation in this way is that, because we have selected one universal reference point, the resulting equation applies at *every point in the flow*.

The following example illustrates how to implement Bernoulli's equation in this manner. It involves flow with a free surface, i.e., a flow with an interface between a liquid and a gas. As in the preceding example, we make use of the fact that the surrounding air impresses atmospheric pressure, p_a, throughout the jet of fluid issuing from the tube.

Example 5.9 *Consider water flowing with uniform velocity, U_1, as shown in the figure. The water enters a uniform-diameter tube at some point below the surface. Determine the velocity of the water leaving the tube at a height z_2 above the surface. Also, determine the pressure in the primary stream of fluid.*



Solution. Clearly, one point where we know everything about the flow is at the free surface very far upstream. Because of the interface with the air, the pressure must be atmospheric so that $p = p_a$. Letting the free surface lie at $z = 0$, the potential energy is zero. Since the flow is uniform far upstream, we also know that the velocity is U_1. Thus, we have

$$p + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho g z = p_a + \frac{1}{2}\rho U_1^2$$

Hence, since $p = p_a$ in the jet of fluid issuing from the tube, we find

$$p_a + \frac{1}{2}\rho U_2^2 + \rho g z_2 = p_a + \frac{1}{2}\rho U_1^2$$

wherefore

$$U_2 = \sqrt{U_1^2 - 2gz_2}$$

Provided we are not too close to either the tube or the bottom, we expect the flow to be uniform with velocity $\mathbf{u} = U_1\,\mathbf{i}$. Then, Bernoulli's equation becomes

$$p + \frac{1}{2}\rho U_1^2 + \rho g z = p_a + \frac{1}{2}\rho U_1^2 \qquad \Longrightarrow \qquad p = p_a - \rho g z$$

Thus, in the moving stream, the pressure satisfies the hydrostatic relation. Close to the tube, the velocity will deviate from the freestream value and, correspondingly, the pressure will depart from the hydrostatic relation. In a real fluid, viscous effects would result in nonuniform velocity near the bottom, which would also cause the pressure to differ from the hydrostatic value.

5.7 Velocity-Measurement Techniques

We can use Bernoulli's equation to infer velocity from a pressure measurement. This is useful because pressure is fundamentally easier to measure than velocity. To understand how this is done, we must first introduce the notion of a **stagnation point**. Then, we discuss two measurement devices known as the **Pitot tube** and the **Pitot-static tube**.

5.7.1 Stagnation Points

Figure 5.5 illustrates ideal (i.e., frictionless) two-dimensional flow past a cylinder. The figure includes several streamlines. As discussed in Section 4.4, these are contours that are everywhere parallel to the flow velocity, which means fluid particles move along the streamlines. Since the contours are parallel to the velocity there is no flow across (normal to) streamlines. Because there is no flow across a solid boundary, the cylinder surface is a streamline. We call the streamline coincident with the x axis upstream and downstream of the cylinder the **dividing streamline**. This streamline splits at the front of the cylinder so that half of the fluid moves over the cylinder and half moves under. The streamlines rejoin at the back of the cylinder.

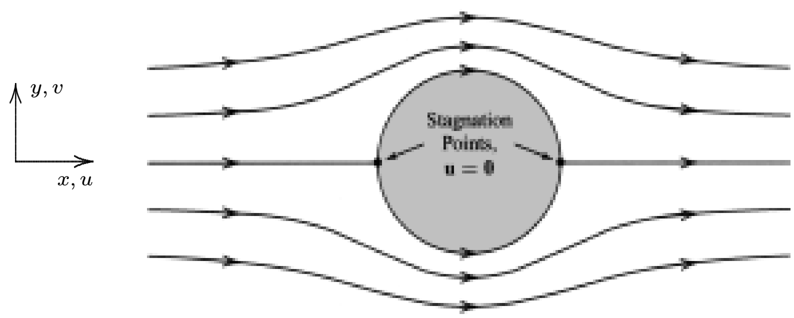


Figure 5.5: *Ideal flow past a cylinder.*

We have a very special situation at these two points. Specifically, we have the intersection of two perpendicular streamlines, namely, the dividing streamline and the cylinder surface. Now, since streamlines are parallel to the velocity, necessarily the vertical velocity component, v, on the dividing streamline must be zero approaching the cylinder. Also, very close to the front of the cylinder we must have zero horizontal velocity, u, on the surface (see Figure 5.5) in order to have flow tangent to the surface. Thus, at the point where the streamlines collide,

$$\mathbf{u} = \mathbf{0} \qquad \text{(Stagnation Point)} \qquad (5.53)$$

So, we have two points on the cylinder where the velocity vanishes, and we refer to these points as **stagnation points**.

5.7.2 Pitot Tube

The **Pitot tube** is one of the simplest devices based on Bernoulli's equation that provides an indirect measurement of velocity. Figure 5.6 illustrates flow in the immediate vicinity of a Pitot tube placed a distance d below the surface in a flowing stream of water. For precise measurements, the tube must have a very small diameter such as that characteristic of a hypodermic needle. As shown, the water fills the Pitot tube up to a point a distance h above the free surface. Because the fluid in the tube cannot move, there must be a stagnation point at the tube's entrance below the surface. As we will now show, this device permits determining the velocity by a simple measurement of the distance the fluid rises above the surface. It is thus the analog, for moving fluids, of the U-Tube manometer discussed in Section 3.4.

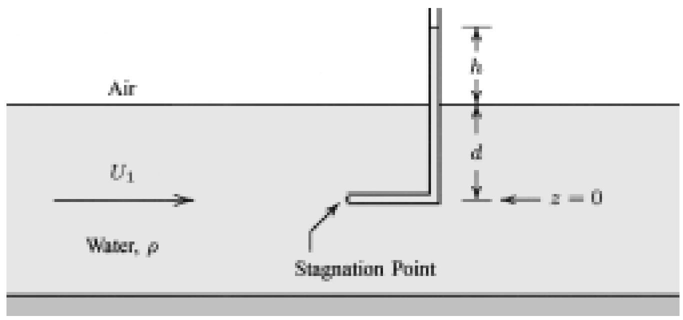


Figure 5.6: *Pitot tube.*

Now, Bernoulli's equation tells us that for this flow we have

$$p + \frac{1}{2}\rho\,\mathbf{u}\cdot\mathbf{u} + \rho g z = \text{constant} \tag{5.54}$$

We can evaluate the constant by selecting a point in the flowfield where the values of pressure, velocity and z are all known. As with the vertical-tube example considered earlier (see Example 5.9), we select a point far upstream of the Pitot tube at the free surface. At this point, the pressure is equal to the atmospheric pressure, p_a, the velocity assumes its freestream value, U_1, and $z = d$ (note that we are choosing the origin to be coincident with the lower portion of the Pitot tube). Hence,

$$\text{constant} = p_a + \frac{1}{2}\rho U_1^2 + \rho g d \tag{5.55}$$

Now, at the top of the tube, which is open to the atmosphere, we know the pressure is p_a, the velocity is zero and $z = d + h$. Hence, applying Bernoulli's equation, we have

$$p_a + \rho g(d+h) = p_a + \frac{1}{2}\rho U_1^2 + \rho g d \tag{5.56}$$

Simplifying, we arrive at the following straightforward relation between the flow velocity and the height of the column of fluid in the Pitot tube.

$$U_1 = \sqrt{2gh} \tag{5.57}$$

While the Pitot tube permits a correlation between the height of a column of fluid and the fluid velocity, it is clearly limited to flows with uniform velocity, and requires a point where velocity and pressure are both known. The device has no provision for flows in which the velocity varies with z, e.g., near the bottom of the channel shown in Figure 5.6 where viscous effects are important.

5.7.3 Pitot-Static Tube

The **Pitot-static tube** is a measuring device based on Bernoulli's equation that can be used for more general velocity distributions. This device makes two separate pressure measurements. The first measurement is done with a standard Pitot tube, which measures the pressure at the tip of the probe. Because this is a stagnation point, the Pitot tube measures the **stagnation pressure**, p_{stag}. The second measurement is at a point downstream of the probe tip sufficiently distant (typically 8 tube diameters) that the flow has returned to its freestream value, U. The pressure at this point is the freestream pressure, also referred to as the **static pressure**, p_{static}. Figure 5.7 schematically depicts a Pitot-static tube.

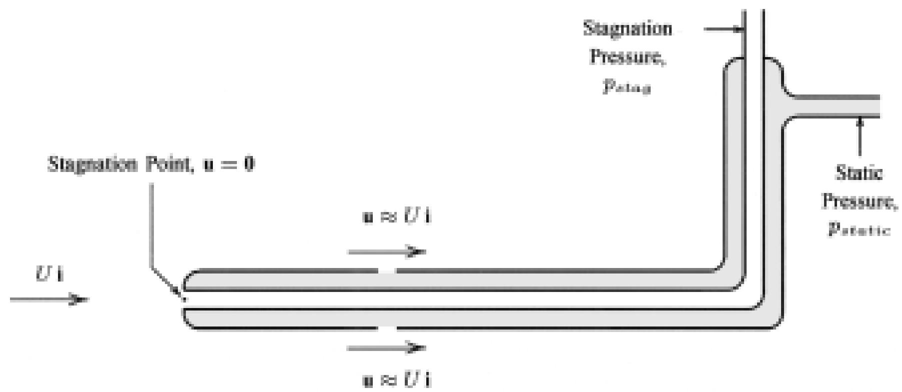


Figure 5.7: *Pitot-static tube.*

Assuming the tube is extremely thin (as it must be to avoid changing the flow), we can ignore the difference in depth of the stagnation point and the static pressure tap. Hence, from Bernoulli's equation, we have

$$p_{\text{stag}} = p_{\text{static}} + \frac{1}{2}\rho U^2 \tag{5.58}$$

That is, the stagnation pressure is the sum of the static pressure and the **dynamic pressure**, $\frac{1}{2}\rho U^2$. Therefore the local velocity is given by

$$U = \sqrt{\frac{2\left(p_{\text{stag}} - p_{\text{static}}\right)}{\rho}} \tag{5.59}$$

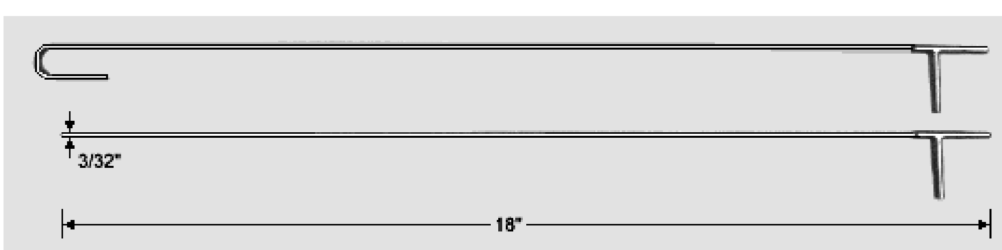


Figure 5.8: *Hand-held Pitot-static tubes used in the automobile racing engine industry. [Photograph courtesy of and © Audie Technology, Inc.]*

Clearly, the Pitot-static tube is not limited to uniform velocity distributions. Furthermore, the device is essentially self calibrating in the sense that no reference pressure or velocity is needed. Although somewhat sensitive to misalignment with flow direction, it is one of the most useful tools in experimental fluid mechanics.

Example 5.10 *A Pitot-static tube is placed in a flow of helium with ρ = 0.16 kg/m^3. The stagnation- and static-pressure taps read 103 kPa and 101 kPa, respectively. What is the velocity of the helium?*

Solution. Using Equation (5.59), the flow velocity is

$$U = \sqrt{\frac{2(103000 - 101000)\ \mathrm{N/m^2}}{0.16\ \mathrm{kg/m^3}}} = 158\ \frac{\mathrm{m}}{\mathrm{sec}}$$

Problems

5.1 Consider steady, compressible flow of a gas through a nozzle. The velocity can be approximated as $\mathbf{u} = U_o(1+x/x_o)\,\mathbf{i}$, where U_o and x_o are constant reference velocity and length, respectively. Determine the density, ρ, if its value at $x = 0$ is ρ_a. At what point does the density fall to 60% of ρ_a?

5.2 For steady flow of a compressible fluid, the velocity vector is

$$\mathbf{u} = u_o \left(\frac{x}{x_o}\right)^2 \mathbf{i}$$

where u_o and x_o are reference velocity and position, respectively. The fluid density is ρ_o at $x = x_o$. Determine the density, ρ, as a function of ρ_o, x and x_o.

5.3 In a one-dimensional, compressible flow, the density decreases exponentially from ρ_a to ρ_b, i.e., $\rho = \rho_a - (\rho_a - \rho_b)\, e^{-t/\tau}$, where τ is a constant of dimensions time. If the velocity at $x = 0$ is $u(0,t) = u_o$, where u_o is constant, determine $u(x,t)$ as a function of u_o, ρ_a, ρ, x, t and τ.

5.4 Appendix D includes the divergence of the velocity in cylindrical and spherical coordinates. Determine the most general form of the velocity for incompressible flow if the following conditions hold.

(a) Axially-symmetric flow with $u_\theta = w = 0$.

(b) Spherically-symmetric flow with $u_\theta = u_\phi = 0$.

5.5 The velocity vector for a flow is

$$\mathbf{u} = \frac{U}{h^3}\left[6x^2 y\,\mathbf{i} + 2x^3\mathbf{j} + 10h^3\mathbf{k}\right]$$

where U and h are constants. Is the flow incompressible? Is the flow irrotational?

5.6 The Cartesian velocity components for a two-dimensional flow are

$$u = \frac{UD^2 y}{(x^2+y^2)^{3/2}}, \qquad v = -\frac{UD^2 x}{(x^2+y^2)^{3/2}}$$

where U and D are constants. Is the flow incompressible? Is the flow irrotational?

5.7 The velocity vector for a flow is

$$\mathbf{u} = \frac{U_o}{L^3}\left[x^2 y\,\mathbf{i} + xy^2\,\mathbf{j} - 4xyz\,\mathbf{k}\right]$$

where U_o and L are constants. Is the flow incompressible? Is the flow irrotational?

5.8 A flowfield has the following velocity vector

$$\mathbf{u} = \frac{x^3 z}{y^2}\mathbf{i} - \frac{3x^2 z}{y}\mathbf{j} - \frac{3x^2 z^2}{y^2}\mathbf{k}$$

where all quantities are dimensionless. Is the flow incompressible? Is the flow irrotational?

5.9 The Cartesian velocity components for a two-dimensional flow are

$$u = \frac{UR^2(y^2-x^2)}{(x^2+y^2)^2}, \qquad v = -\frac{2UR^2 xy}{(x^2+y^2)^2}$$

where U and R are constants. Is the flow incompressible? Is the flow irrotational?

5.10 The velocity for a two-dimensional flow in cylindrical coordinates is

$$\mathbf{u} = U\left(\frac{r}{R}\right)\cos\theta\,\mathbf{e}_r - 2U\left(\frac{r}{R}\right)\sin\theta\,\mathbf{e}_\theta$$

where U and R are constants. Is the flow incompressible? Is the flow irrotational?

5.11 The velocity for a two-dimensional flow in cylindrical coordinates is

$$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \qquad u_\theta = U\frac{R}{r} - U\left(1 + \frac{R^2}{r^2}\right)\sin\theta$$

where U and R are constants. Is the flow incompressible? Is the flow irrotational?

5.12 The y component of the velocity vector for a two-dimensional, incompressible, irrotational flow is $v(x,y) = Bx/(x^2+y^2)$, where B is a constant. Determine the x component, $u(x,y)$.

5.13 The y component of the velocity vector for a two-dimensional, incompressible, irrotational flow is $v(x,y) = -U(y/h-1)$ where U and h are constants. Determine the x component, $u(x,y)$.

5.14 The x component of the velocity vector for a two-dimensional, incompressible, irrotational flow is

$$u(x,y) = U\left[1 - e^{-\lambda x}\cos\lambda y\right]$$

where U and λ are constant velocity and inverse length scales, respectively. Determine the y component of the velocity vector, $v(x,y)$, assuming there is a stagnation point at $x = y = 0$.

5.15 The x component of the velocity vector for a two-dimensional, incompressible, irrotational flow is $u(x,y) = Cxy$ where C is a constant. Determine the y component, $v(x,y)$.

5.16 The circumferential component of the velocity vector for a two-dimensional, incompressible, irrotational flow is $u_\theta(r,\theta) = -3Ar^2\sin 3\theta$ where A is a constant. Determine the radial component, $u_r(r,\theta)$.

5.17 The radial component of the velocity vector for a two-dimensional, incompressible, irrotational flow is $u_r(r,\theta) = (S/r^2)\cos\theta$ where S is a constant. Determine the circumferential component, $u_\theta(r,\theta)$.

5.18 A flowfield has the velocity vector $\mathbf{u} = Ar\cos\theta\,\mathbf{e}_r - r\sin\theta\,\mathbf{e}_\theta$, where A is a constant and all quantities are dimensionless.

(a) Is there any value of A for which this flow is irrotational?

(b) Is there any value of A for which this flow is incompressible?

5.19 An unsteady flow has the velocity vector $\mathbf{u} = xe^{-t/\tau}\mathbf{i} + Cye^{-t/\tau}\mathbf{j}$, where C and τ are constants and all quantities are dimensionless. The flow is incompressible and irrotational. Find the values of C and τ necessary to guarantee these conditions.

5.20 Consider unsteady flow of an incompressible fluid with negligible body forces. The velocity vector is $\mathbf{u} = U\,\mathbf{i} + Ue^{-t/\tau}\mathbf{j}$, where U and τ are constants. Determine the pressure, $p(x,y,z,t)$, for this flow.

5.21 Consider an unsteady flow in an incompressible fluid in which gravitational effects are important. The velocity and gravitational vectors are $\mathbf{u} = U\,\mathbf{i} + U\cosh[\kappa(x-Ut)]\mathbf{k}$ and $\mathbf{g} = -g\,\mathbf{k}$, where U and κ are constants and g is gravitational acceleration. Determine the pressure, $p(x,y,z,t)$, for this flow.

5.22 Consider a pipe whose cross-sectional area is $A(x) = A_o F(x)$, where A_o is the area at $x = 0$, $F(0) = 1$ and $F(x)$ is an obscure function you've never heard of. Assume the flow is inviscid, incompressible, can be approximated as one dimensional and that body forces are negligible. Using the fact that mass flux, $\dot{m} = \rho u A$, is constant and $p(0) = p_a$, compute the pressure throughout the pipe. How does $p(x)$ vary with increasing area? How does it vary with decreasing area? Explain your results.

5.23 The average velocity of water in a nozzle increases from $u_1 = 4$ m/sec to $u_2 = 10$ m/sec. Assuming the average velocity varies linearly with distance along the nozzle, x, and that the length of the nozzle is $\ell = 1$ m, estimate the pressure gradient, dp/dx, at a point midway through the nozzle. The density of water is $\rho = 1000$ kg/m^3. You may assume the flow can be approximated as one dimensional.

5.24 The velocity vector for a steady, incompressible flow is $\mathbf{u} = (U/L^2)(yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k})$, where U and L are constant velocity and length scales, respectively. The fluid density is ρ and the pressure at $x = y = z = 0$ is p_o. Verify that this flow is irrotational and incompressible. Also, using Euler's equation, determine the pressure as a function of ρ, U, L, p_o, x, y and z. Assume there are no body forces acting. Compare your result with the pressure according to Bernoulli's equation.

5.25 We wish to analyze an incompressible, two-dimensional flow with velocity vector $\mathbf{u} = Ax\,\mathbf{i} - Ay\,\mathbf{j}$, where A is a constant and body forces are negligible.

(a) Does this velocity field satisfy the continuity equation?

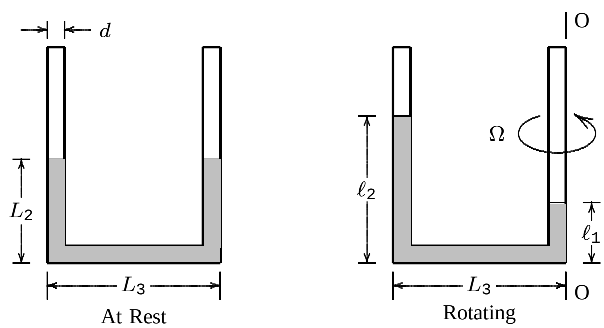
(b) Is the flow irrotational?

(c) If the flow is inviscid, what is the pressure, $p(x,y)$, if $p(0,0) = p_t$?

5.26 The constant-diameter U-tube shown rotates about axis O-O at angular velocity Ω.

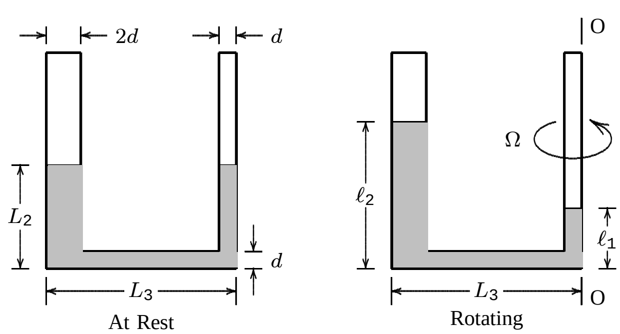
(a) Determine the new positions of the water surfaces, ℓ_1 and ℓ_2. Neglect the diameter of the U-tube in your computations and assume the tubes are open to the atmosphere.

(b) What does your answer for Part (a) predict for rotation rates in excess of the critical value defined by $\Omega_{crit} = 2\sqrt{gL_2}/L_3$? Explain how to reformulate the problem with a diagram of the fluid in the U-tube when $\Omega > \Omega_{crit}$.



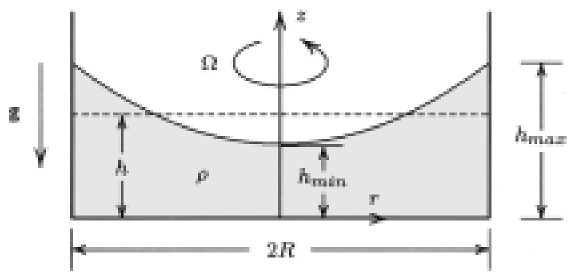
Problem 5.26

5.27 Determine the new positions of the water surfaces, ℓ_1 and ℓ_2 when the U-tube shown rotates about axis O-O at angular velocity Ω. Assume the diameter of the thickest part of the U-tube, $2d$, is very small compared to L_3, and that the tubes are open to the atmosphere.



Problem 5.27

5.28 Fluid in a cylindrical tank of radius R rotates about the z axis with angular velocity Ω. The fluid has been rotating for a time sufficient to establish rigid-body rotation. The initial fluid level (indicated by the dashed line) is h, the fluid density is ρ and atmospheric pressure is p_a. As shown in Subsection 5.5.1, the equation of the free surface is $z = \zeta(r) = h_{min} + \frac{1}{2}\Omega^2 r^2/g$. Appealing to mass conservation, compute h_{max} and h_{min} as functions of h, Ω, R and g.

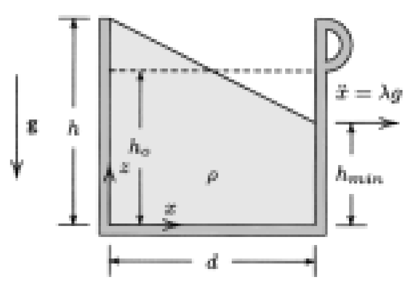


Problems 5.28, 5.29, 5.30

5.29 Fluid in a cylindrical tank of radius R rotates about the z axis with angular velocity Ω. The fluid has been rotating for a time sufficient to establish rigid-body rotation. The initial fluid level (indicated by the dashed line) is h, the fluid density is ρ and atmospheric pressure is p_a. As shown in Subsection 5.5.1, the equation of the free surface is $z = \zeta(r) = h_{min} + \frac{1}{2}\Omega^2 r^2/g$. Appealing to mass conservation, find the rotation rate for which the center of the container just becomes exposed, i.e., $h_{min} = 0$.

5.30 A cylindrical tank partially filled with a liquid of density ρ rotates about the z axis with angular velocity Ω. The tank is open to the atmosphere and has been rotating for a time sufficient to establish rigid-body rotation. Determine the shape of the free surface on the plane passing through the axis of rotation if the minimum and maximum depths are $h_{min} = 0.08R$ and $h_{max} = 0.4R$. Also, determine the angular-rotation rate, Ω, as a function of g and R.

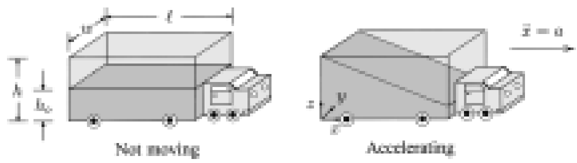
5.31 Imagine you are rushing to the university to avoid being late for your fluid-mechanics class. Your coffee cup is resting next to you in your car. In your haste to get to class, you accelerate at λ g's, i.e., your acceleration is $a = \lambda g$ where λ is a constant. What is the maximum height, h_o, to which the cup can be filled to avoid spilling any coffee? Assume the cup is a cylinder of height h = 10 cm and diameter d = 8 cm. **HINT:** To conserve mass in this geometry, necessarily $h_o = \frac{1}{2}(h + h_{min})$. As a percentage, determine how full the cup can be if you are driving your Volkswagen Bug ($\lambda = 1/6$) or your Corvette ($\lambda = 2/5$).



Problems 5.31, 5.32

5.32 Imagine you are rushing to the university to avoid being late for your fluid-mechanics class. Your coffee cup is resting next to you in your car. In your haste to get to class, you accelerate at λ g's, i.e., your acceleration is $a = \lambda g$ where λ is a constant. What is the maximum value of λ possible to avoid spilling if the cup is initially 75% full? Assume the cup is a cylinder of height h = 4 inches and diameter d = 3.5 inches. **HINT:** To conserve mass in this geometry, necessarily $h_o = \frac{1}{2}(h + h_{min})$.

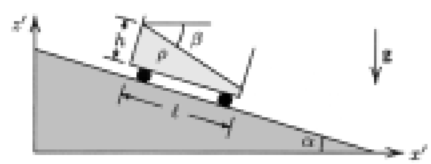
5.33 A truck carries a tank of water that is open at the top with length ℓ, width w and depth h. Assuming the driver will not accelerate the truck at a rate greater than a, what is the maximum depth, h_o, to which the tank may be filled to prevent spilling any water? Assume constant acceleration, and note that the pressure is constant and equal to its atmospheric value at the free surface.



Problems 5.33, 5.34

5.34 A truck carries a tank of water that is open at the top with length ℓ, width w and depth h. If the truck is half full and $\ell = 5h$, what is the maximum acceleration, a, that can be sustained without spilling any water? Assume constant acceleration, and note that the pressure is constant and equal to its atmospheric value at the free surface.

5.35 A small car containing an incompressible fluid of density ρ is rolling down an inclined plane. Determine the location and value of the maximum pressure in the car. Ignore any friction in the wheels. **HINT:** Use a coordinate system with x and z parallel to and normal to the inclined plane, respectively.



Problems 5.35, 5.36

5.36 A small car containing an incompressible fluid of density ρ is rolling down an inclined plane. Show that the free surface is planar and determine the angle it makes with the horizontal, β. Ignore any friction in the wheels. **HINT:** Do your work in a coordinate system for which x and z are parallel to and normal to the inclined plane, respectively.

5.37 In terms of natural coordinates (see Appendix D, Section D.5), the s and n components of Euler's equation are

$$\rho\, u \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s} - \rho \frac{\partial \mathcal{V}}{\partial s} \qquad \text{and} \qquad -\rho \frac{u^2}{\mathcal{R}} = -\frac{\partial p}{\partial n} - \rho \frac{\partial \mathcal{V}}{\partial n}$$

As discussed in Section 5.6, integrating the streamwise, or s, component yields

$$p + \frac{1}{2}\rho\, u^2 + \rho \mathcal{V} = F(n)$$

where $F(n)$ is constant along a streamline (because n = constant on a streamline). Verify that, with $\omega = u/\mathcal{R} + \partial u/\partial n$ denoting the vorticity,

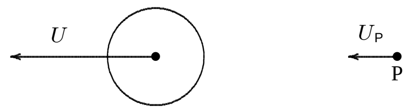
$$F'(n) = \rho \omega u$$

5.38 For hurricanes, the Coriolis acceleration is much larger than the convective acceleration. Reference to Appendix D, Section D.4 shows that the Euler equation simplifies to

$$2\rho\, \mathbf{\Omega} \times \mathbf{u} = -\nabla \hat{p}$$

where $\hat{p}$ is a reduced pressure that includes the centrifugal acceleration and $\mathbf{\Omega}$ is Earth's angular velocity. Based on this equation, explain why hurricanes rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. **HINT:** To simplify your explanation, consider hurricanes centered at the north pole and the south pole.

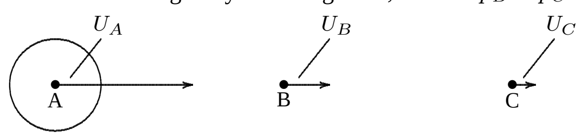
5.39 A cylinder moves at constant velocity U through water of density ρ = 1000 kg/m^3. The difference between the pressure at the front stagnation point on the cylinder and at Point P in the cylinder's wake is $p_t - p_\mathsf{P}$ = 4.5 kPa. If the velocity at Point P is U_P = 5 m/sec, what is U?



Problems 5.39, 5.40

5.40 A cylinder moves at constant velocity U = 9 m/sec through water of density ρ = 1000 kg/m^3. The difference between the pressure at the front stagnation point on the cylinder and at Point P in the cylinder's wake is $p_t - p_\mathsf{P}$ = 18 kPa. What is the velocity at Point P?

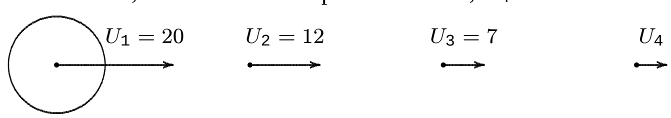
5.41 Body A travels through water at a constant speed of U_A = 20 ft/sec. Velocities at points B and C are induced by the moving body and have magnitudes of U_B = 7 ft/sec and U_C = 3 ft/sec. If the density of water is 1.94 slug/ft^3 and effects of gravity can be ignored, what is $p_B - p_C$ in psi?



Problems 5.41, 5.42

5.42 Body A travels through water at a constant speed of U_A = 15 m/sec. Velocities at points B and C are induced by the moving body and have magnitudes of U_B = 4 m/sec and U_C = 2 m/sec. If the density of water is 1000 kg/m^3 and effects of gravity can be ignored, what is $p_B - p_C$ in kPa?

5.43 An object is traveling through water at a speed U_1 = 20 m/sec. The flow speeds at Points 2 and 3 are U_2 = 12 m/sec and U_3 = 7 m/sec. If the pressure difference between Points 3 and 4 is half the difference between Points 2 and 3, what is the flow speed at Point 4, U_4?



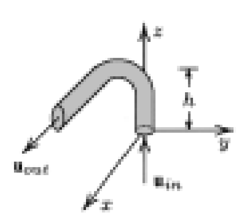
Problem 5.43

5.44 Reference to Appendix D, Section D.4 shows that for inviscid flow in a coordinate system rotating about the z axis with angular velocity $\mathbf{\Omega} = \Omega\mathbf{k}$, the momentum equation assumes the following form.

$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\frac{\nabla' p}{\rho} - \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}' - 2\mathbf{\Omega} \times \mathbf{u}'$$

The last two terms on the right-hand side of this equation are the centrifugal and Coriolis forces, respectively. Taking advantage of the results developed in Section 5.5.2, determine the form this equation assumes under a Galilean transformation. Is this equation Galilean invariant?

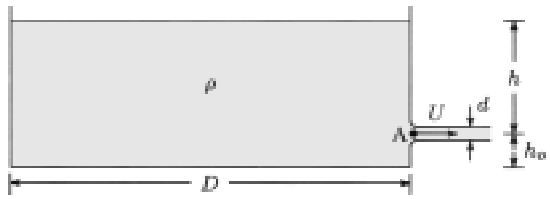
5.45 The watering tube shown has a 90$^\mathrm{o}$ bend and an outlet velocity given by $\mathbf{u}_{out} = 5\mathbf{i} - 2\mathbf{j}$ ft/sec. The pressure is atmospheric at the outlet, the fluid enters the tube at a speed of 7 ft/sec and the tube is h = 10 in high. What is the pressure at the inlet, p_{in}? Assume the flow is steady and irrotational.



Problem 5.45

5.46 Determine the maximum pressure on your hand when you hold it out the window of your automobile on a day when the ambient pressure is 1 atm and the temperature is 68° F. Assuming the conditions required for Bernoulli's equation to hold are satisfied, compute the maximum pressure (in atm) when you are cruising along a highway at 70 mph and diving your Indy 500 racer at 200 mph.

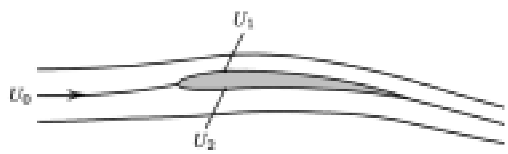
5.47 The velocity in the outlet pipe from a large reservoir of depth $h = 10$ m is U = 11 m/sec. Due to the rounded entrance to the pipe, the flow can be assumed to be irrotational. Also, the reservoir is so large that the flow is essentially steady. With these conditions, what is the pressure at point A as a function of the fluid density, ρ, gravitational acceleration, g, atmospheric pressure, p_a, as well as U and h? Determine the value of $p - p_a$ in kPa for water whose density is $\rho = 1000$ kg/m^3.



Problems 5.47, 5.48

5.48 The velocity in the outlet pipe from a large reservoir of depth $h = 25$ ft is U = 30 ft/sec. Due to the rounded entrance to the pipe, the flow can be assumed to be irrotational. Also, the reservoir is so large that the flow is essentially steady. With these conditions, what is the pressure at point A as a function of the fluid density, ρ, gravitational acceleration, g, atmospheric pressure, p_a, as well as U and h? Determine the value of $p - p_a$ in psi for water whose density is $\rho = 1.94$ slug/ft^3.

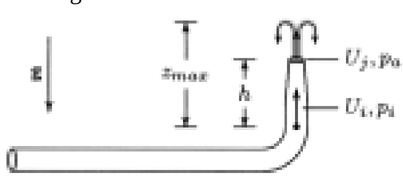
5.49 The streamlines for inviscid flow past an airfoil are as shown. Freestream velocity, U_0, is 90 m/sec and fluid density, ρ, is 1.20 kg/m^3. Assuming Bernoulli's equation applies, what is the pressure difference, $p_2 - p_1$ (in kPa), at points where the velocities are U_1 = 100 m/sec and U_2 = 80 m/sec?



Problems 5.49, 5.50

5.50 The streamlines for inviscid flow past an airfoil are as shown. Freestream velocity, U_0, is 120 ft/sec and fluid density, ρ, is 0.00234 slug/ft^3. Assuming Bernoulli's equation applies, what is the pressure difference, $p_2 - p_1$ (in psi), at points where the velocities are U_1 = 180 ft/sec and U_2 = 110 ft/sec?

5.51 The figure depicts incompressible flow through a pipe and nozzle that emits a vertical jet. The flow is steady, irrotational, has density ρ and the only body force is gravity. What is the velocity of the jet at the nozzle exit, U_j? To what height, z_{max}, will the jet of fluid rise? Express your answers in terms of ρ, g, h, U_i and $p_i - p_a$. Determine the numerical values of U_j and z_{max} if $p_i - p_a$ = 60 kPa, h = 1 m, U_i = 4 m/sec and ρ = 1000 kg/m^3.



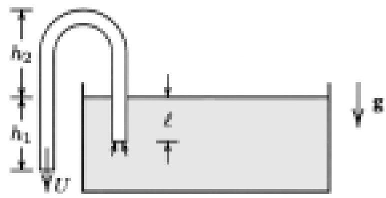
Problems 5.51, 5.52

5.52 The figure depicts incompressible flow through a pipe and nozzle that emits a vertical jet. The flow is steady, irrotational, has density ρ and the only body force is gravity. If the jet of fluid rises to a height $z_{max} = 2h$, what is the value of $p_i - p_a$ as a function of ρ, g, h and U_i? Determine the numerical value of $p_i - p_a$ in psf if ρ = 1.98 slug/ft^3, h = 4 ft and U_i = 12 ft/sec.

5.53 A siphon tube of constant diameter d is attached to a large tank as shown. You can assume the flow is quasi-steady, incompressible, irrotational and that gravity is the only body force.

(a) Find the outlet velocity, U, and the minimum pressure in the siphon tube, p_{min}, as functions of gravitational acceleration, g, fluid density, ρ, atmospheric pressure, p_a, and the distances ℓ, h_1 and h_2. **HINT:** Use the fact that the mass flux through the tube, $\dot{m} = \frac{\pi}{4}\rho|\mathbf{u}|d^2$, is constant.

(b) Calculate U and p_{min} for h_1 = 4 m, h_2 = 4 m, ℓ = 3 m, ρ = 1000 kg/m^3 and p_a = 1 atm.



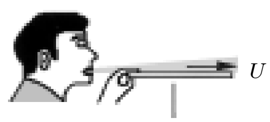
Problems 5.53, 5.54

5.54 A siphon tube of constant diameter d is attached to a large tank as shown. You can assume the flow is quasi-steady, incompressible, irrotational and that gravity is the only body force.

(a) Find the outlet velocity, U, and the minimum pressure in the siphon tube, p_{min}, as functions of gravitational acceleration, g, fluid density, ρ, atmospheric pressure, p_a, and the distances ℓ, h_1 and h_2. **HINT:** Use the fact that the mass flux through the tube, $\dot{m} = \frac{\pi}{4}\rho|\mathbf{u}|d^2$, is constant.

(b) Calculate U and p_{min} for h_1 = 5 ft, h_2 = 3 ft, ℓ = 3 ft, ρ = 1.94 slug/ft^3 and p_a = 1 atm.

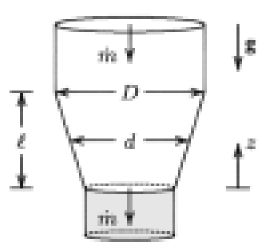
5.55 An eager fluid-mechanics student blows a stream of air over one side of a 21 cm by 29.7 cm sheet of paper. The density of air is ρ = 1.20 kg/m^3. Assuming the stream blows over the entire surface at a velocity U = 1.20 m/sec and the paper is in a horizontal position, what is the weight of the paper in Newtons? What is the pressure difference between the upper and lower surfaces of the paper?



Problems 5.55, 5.56

5.56 An eager fluid-mechanics student blows a stream of air over one side of an $8\frac{1}{2}$ inch by 11 inch sheet of paper weighing W = 0.01 lb. The density of air is ρ = 0.0024 slug/ft^3. Assuming the stream blows over the entire surface, what velocity, U, is required to support the weight of the sheet of paper in a horizontal position? What is the pressure difference between the upper and lower surfaces of the paper?

5.57 The figure shows a downward-facing round nozzle that is attached to a hose through which an incompressible fluid of density ρ flows. The hose diameter is D and the (constant) mass flux is $\dot{m} = \frac{\pi}{4}\rho|w|d^2$, where d is nozzle diameter and w is vertical velocity. Treating the flow as one-dimensional, determine how the nozzle diameter must vary with z in order to have atmospheric pressure, p_a, throughout the nozzle. Assume the flow is steady, irrotational and the only body force acting is gravity.



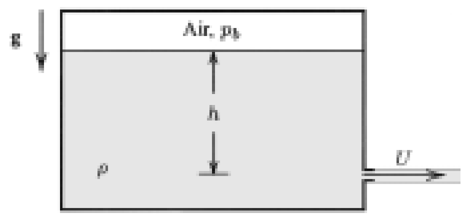
Problems 5.57, 5.58

5.58 The figure shows a downward-facing round nozzle that is attached to a hose through which an incompressible fluid of density ρ flows. The hose diameter is D and the (constant) mass flux is $\dot{m} = \frac{\pi}{4}\rho|w|d^2$, where d is nozzle diameter and w is vertical velocity. Treating the flow as one-dimensional, determine how the nozzle diameter must vary with z in order to have $dp/dz = -\frac{1}{4}\rho g$ throughout the nozzle. The quantity g is gravitational acceleration. Assume the flow is quasi-steady, irrotational and the only body force acting is gravity.

5.59 A large closed tank is pressurized as shown. A jet of fluid issues from a small hole. Assume the flow is incompressible, irrotational, quasi-steady and the only body force acting is gravity.

(a) Determine the pressure, p_b, required to increase the jet velocity by 50% relative to the value realized for atmospheric pressure, p_a, in the upper chamber.

(b) Compute the value of p_b in atm when h = 10 m and ρ = 998 kg/m^3. What is the jet velocity, U, for this pressure?



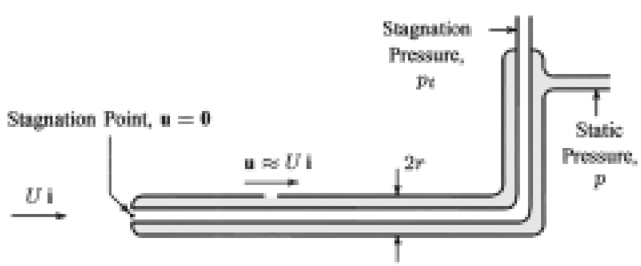
Problems 5.59, 5.60

5.60 A large closed tank is pressurized as shown. A jet of fluid issues from a small hole. Assume the flow is incompressible, irrotational, quasi-steady and the only body force acting is gravity.

(a) Determine the pressure, p_b, required to triple the jet velocity relative to the value realized for atmospheric pressure, p_a, in the upper chamber.

(b) Compute the value of p_b in atm when h = 18 ft and ρ = 1.99 slug/ft^3. What is the jet velocity, U, for this pressure?

5.61 A Pitot-static tube is placed in a flow of air with ρ = 1.20 kg/m^3. The stagnation- and static-pressure taps read 103.16 kPa and 101 kPa, respectively. What is the velocity of the air? If the velocity changes to 80 m/sec and the static pressure is unchanged, what is the corresponding stagnation pressure?

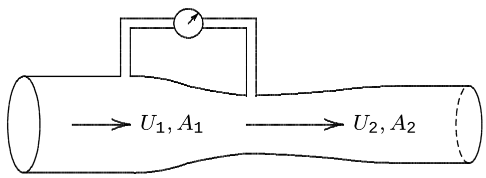


Problems 5.61, 5.62, 5.63

5.62 Consider a poorly designed Pitot-static tube with a single static-pressure hole at the top of the tube as shown. If the tube radius is r, develop a formula for the true velocity, U_{true}, as a function of gravitational acceleration, g, radius, r, and the velocity inferred from $U = \sqrt{2(p_t - p)/\rho}$. If r = 5 mm, determine the percentage error in velocity for an indicated value of U = 1, 10 and 100 m/sec.

5.63 A Pitot-static tube is placed in a flow of helium with $\rho = 3.2 \cdot 10^{-4}$ slug/ft^3. The static-pressure tap reads 1725 psf and the flow velocity is 300 ft/sec. What is the stagnation pressure? If the stagnation pressure changes to 1760 psf and the static pressure is unchanged, what is the corresponding velocity?

5.64 A *Venturi meter* is a device used to measure fluid velocities and flow rates for incompressible, steady flow. As shown, the pressure is measured at two sections of a pipe with different cross-sectional areas. A straightforward derivation shows that the volume-flow rate is $Q = A_2\sqrt{2(p_1 - p_2)/[\rho(1 - A_2^2/A_1^2)]}$, where p is pressure, ρ is density and A is cross-sectional area. Consider a Venturi meter that has A_1 = 6 ft^2 and A_2 = 5 ft^2. If the volume-flow rate is Q = 50 ft^3/sec, what is the pressure difference, $p_1 - p_2$, if the fluid flowing is air, water or mercury?



Problems 5.64, 5.65, 5.66

5.65 A *Venturi meter* is a device used to measure fluid velocities and flow rates for incompressible, steady flow. As shown, the pressure is measured at two sections of a pipe with different cross-sectional areas. A straightforward derivation shows that the volume-flow rate is $Q = A_2\sqrt{2(p_1 - p_2)/[\rho(1 - A_2^2/A_1^2)]}$, where p is pressure, ρ is density and A is cross-sectional area. Consider a Venturi meter that has A_1 = 0.1 m^2 and A_2 = 0.075 m^2. The attached pressure gage can accurately measure pressures no smaller than 0.1 Pa. If air of density ρ = 1.20 kg/m^3 is flowing, what is the minimum flow rate that can be accurately measured?

5.66 A *Venturi meter* is a device used to measure fluid velocities and flow rates for incompressible, steady flow. As shown, the pressure is measured at two sections of a pipe with different cross-sectional areas. You may assume the flow is irrotational and that effects of body forces can be ignored. Noting that for steady flow the mass-flow rate, $\dot{m}$, is constant and equal to $\rho U A$, where ρ is density, U is average velocity and A is cross-sectional area, verify that $\dot{m}$ is

$$\dot{m} = \rho A_2 \sqrt{\frac{2\,(p_1 - p_2)}{\rho\,(1 - A_2^2/A_1^2)}}$$

5.67 A *barotropic fluid* is one for which the pressure is a function only of density, i.e., $p = p(\rho)$. For a steady, irrotational flow of a barotropic fluid with a conservative body force $\mathbf{f} = -\nabla\mathcal{V}$, derive the following replacement for Bernoulli's equation.

$$\int \frac{dp}{\rho} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \mathcal{V} = \text{constant}$$

HINT: Introduce a function $F(\rho)$ defined by

$$\frac{\nabla p}{\rho} = \frac{1}{\rho}\frac{dp}{d\rho}\nabla\rho = F'(\rho)\nabla\rho$$

5.68 For incompressible, irrotational flows, even when the flow is unsteady, the velocity vector, **u**, can be written as $\mathbf{u} = \nabla\phi$, where $\phi(x, y, z, t)$ is the *velocity potential*. Beginning with the vector identity

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - \mathbf{u}\times(\nabla\times\mathbf{u})$$

derive an unsteady-flow replacement for Bernoulli's equation. Assume a conservative body force is present so that $\mathbf{f} = -\nabla\mathcal{V}$.